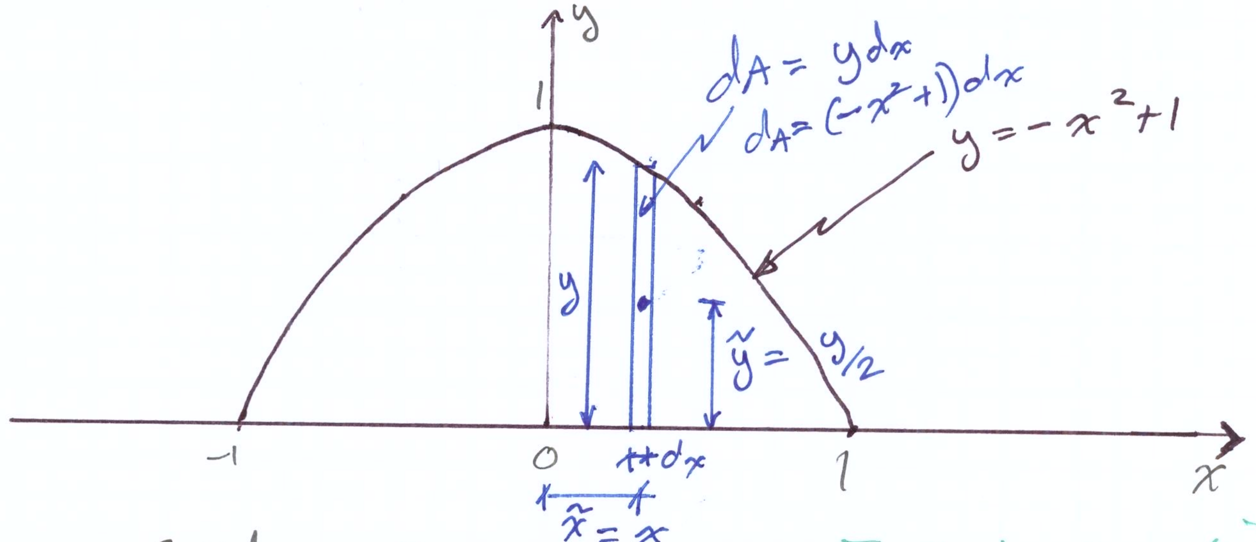
Find the centroid and moment of inertia (I_x, I_y) about the centroidal axes.



$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA}$$

{Note: we know $\bar{x} = 0$ by symmetry → we will calculate it anyway}

$$\bar{x} = \frac{\int_{-1}^{1} x\,(y\,dx)}{\int_{-1}^{1} y\,dx} = \frac{\int_{-1}^{1} x(-x^2+1)\,dx}{\int_{-1}^{1} (-x^2+1)\,dx} = \frac{\left[-\frac{x^4}{4} + \frac{x^2}{2}\right]\Big|_{-1}^{1}}{\left[-\frac{x^3}{3} + x\right]_{-1}^{1}}$$

$$\bar{x} = \frac{\left(-\frac{(1)^4}{4} + \frac{(1)^2}{2}\right) - \left(-\frac{(-1)^4}{4} + \frac{(-1)^2}{2}\right)}{\left(-\frac{(1)^3}{3} + 1\right) - \left(-\frac{(-1)^3}{3} + (-1)\right)} = \frac{0}{\left[-\frac{2}{3} + 2\right]}$$

$$\bar{x} = \frac{0}{\underbrace{4/3}_{A = \int_A dA = \boxed{4/3}}}$$

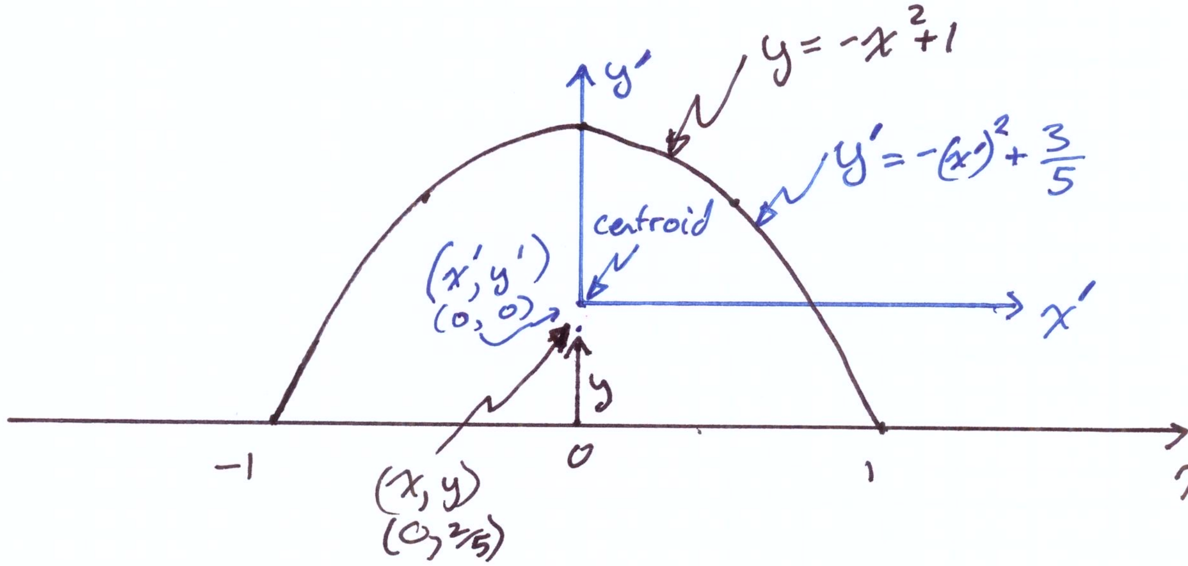
$$\boxed{\bar{x} = 0}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_A (y/2)\,dA}{\int_A dA} = \frac{\int_A \left(\frac{-x^2+1}{2}\right) dA}{\int_A dA} = \frac{\int_{-1}^{1} \left(\frac{-x^2+1}{2}\right)(-x^2+1)\,dx}{\int_A dA}$$

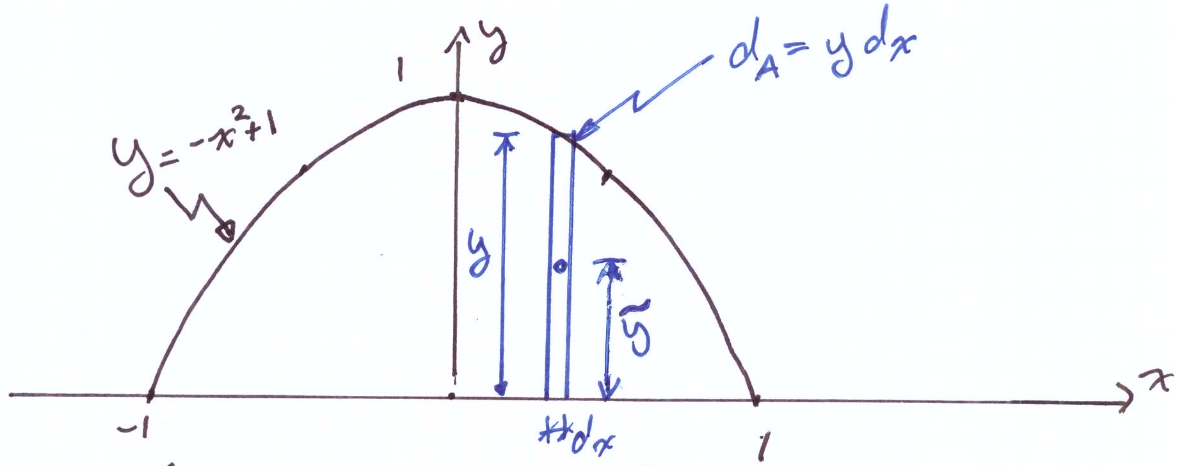
$$\bar{y} = \frac{\frac{1}{2}\int_{-1}^{1} (x^4 - 2x^2 + 1)\,dx}{4/3} = \frac{\frac{1}{2}\left[\frac{x^5}{5} - \frac{2x^3}{3} + x\right]\Big|_{-1}^{1}}{4/3}$$

$$\bar{y} = \frac{3}{8}\left[\left(\frac{1}{5} - \frac{2}{3} + 1\right) - \left(\frac{-1}{5} + \frac{2}{3} - 1\right)\right] = \frac{3}{8}\left[\frac{8}{15} + \frac{8}{15}\right]$$

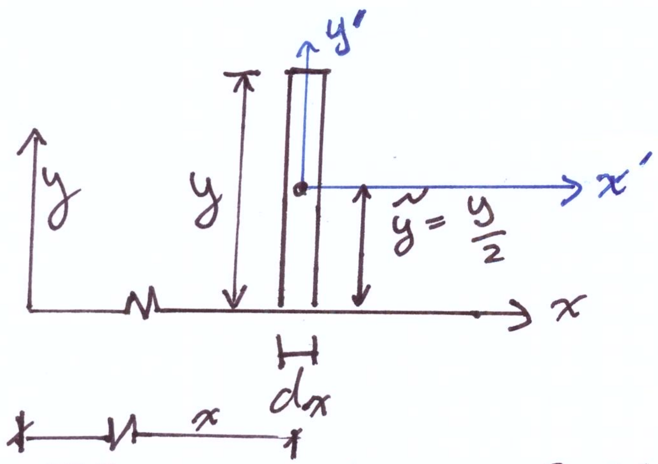
$$\boxed{\begin{aligned} \bar{y} &= \frac{2}{5} = 0.4\,m \\ \bar{x} &= 0 \end{aligned}}$$



- 1st we will find I_x, I_y about the origional axes



- for each strip, we have



$$d\bar{I}_{x'} = \frac{1}{12}(d_x)(y)^3$$

$$dA\tilde{y}^2 = \left(\frac{y}{2}\right)^2 (y)d_x$$

- If we take $\int_A x^2 d_A$ we get I_y since all area strips are parallel to the y-axis and therefore all areas have the same moment arm

- Since there are variable moment arms in the strips about the x-axis, we must apply parallel axis theorem

$$\boxed{dI_x = d\bar{I}_{x'} + dA\tilde{y}^2}$$

- **Find I_x**

$$dI_x = d\bar{I}_x' + dA\tilde{y}^2$$

$$dI_x = \underbrace{\frac{1}{12}(dx)y^3}_{d\bar{I}_x'} + \underbrace{\left(\frac{y}{2}\right)^2 (y)\,dx}_{dA\tilde{y}^2}$$

$$dI_x = \frac{1}{3}y^3\,dx$$

$$I_x = \int_{-1}^{1} dI_x = \int_{-1}^{1}\frac{1}{3}y^3\,dx = \int_{-1}^{+1}\frac{(-x^2+1)^3}{3}\,dx$$

$$I_x = \frac{1}{3}\int_{-1}^{1}(-x^6+3x^4-3x^2+1)\,dx = \frac{1}{3}\left[\frac{-x^7}{7}+\frac{3x^5}{5}-x^3+x\right]\Bigg|_{-1}^{1}$$

$$I_x = \frac{1}{3}\left[\frac{-1}{7}+\frac{3}{5}-1+1\right]-\left[\frac{1}{7}-\frac{3}{5}+1-1\right] = \boxed{I_x = \frac{32}{105} = 0.30476}$$

- **Find I_y**

$$I_y = \int_A x^2\,dA = \int_{-1}^{1}x^2(-x^2+1)\,dx = \int_{-1}^{1}\left[-x^4+x^2\right]dx$$

$$I_y = \left[\frac{-x^5}{5}+\frac{x^3}{3}\right]\Bigg|_{-1}^{1} = \left[-\frac{1}{5}+\frac{1}{3}\right]-\left[\frac{1}{5}-\frac{1}{3}\right]$$

$$\boxed{I_y = 4/15 = 0.26667}$$

- Now we have moments of inertia about axes that are parallel to our centroidal axes, we can find moments of inertia about centroid axes using parallel axis theorem.

$$I_x = \bar{I}_x' + Ad_y^2$$

$$I_y = \bar{I}_y' + Ad_x^2$$

I_x is about x
$\bar{I}_x'$ is about x'

I_y is about y
$\bar{I}_y'$ is about y'

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$\therefore$ If we want $\bar{I}_x'$, $\bar{I}_y'$

$$\boxed{\begin{aligned}\bar{I}_x' &= I_x - Ad_y^2 \\ \bar{I}_y' &= I_y - Ad_x^2\end{aligned}}$$

$$\bar{I}_x' = \frac{32}{105} - \left(\frac{4}{3}\right)\left(\frac{2}{5}\right)^2 = 16/175$$

$$\boxed{\bar{I}_x' = \frac{16}{175} = 0.091429}$$

$$\bar{I}_y' = \frac{4}{15} - \left(\frac{4}{3}\right)(0)^2 = 4/15$$

$$\boxed{\bar{I}_y' = 4/15 = 0.26667}$$